Black-Scholes-Like Derivative Pricing With Tsallis Non-extensive Statistics

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Abstract

We recently showed that the S&P500 stock market index is well described by Tsallis non-extensive statistics and nonlinear Fokker-Planck time evolution. We argued that these results should be applicable to a broad range of markets and exchanges where anomalous diffusion and 'heavy' tails of the distribution are present. In the present work we examine how the Black-Scholes derivative pricing formula is modified when the underlying security obeys non-extensive statistics and Fokker-Planck time evolution. We answer this by recourse to the underlying microscopic Ito-Langevin stochastic differential equation of the non-extensive process.

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For certain stochastic systems there is an interesting connection between statistics and dynamics. A family of nonlinear Fokker-Planck time-evolution equations turns out to be solved by probability distributions which are characterized by Tsallis non-extensive statistics[1, 2, 3]. Recently we exploited this connection to analyze the dynamics of the S&P500 stock index[4], showing that price-change distributions had both non-extensive form and Fokker-Planck time evolution. We argued that the results should be applicable to the broad range of markets and exchanges characterized by anomalous (super) diffusion and 'heavy' distribution tails[5, 6, 7]. In this paper we now investigate how the Black-Scholes derivative pricing formula[8] is modified when the underlying security is described by non-extensive statistics. This is based on an analysis of the microscopic Ito-Langevin stochastic differential equation underlying the macroscopic nonlinear Fokker-Planck equation[2, 9, 10, 11].

I. NON-EXTENSIVE STATISTICS AND TIME EVOLUTION

We begin by summarizing the probability distribution function (PDF) $P(S,t)$ which is obtained using non-extensive statistics. Denote the value of a security at a trading time τ by price(τ). In the following we will take prices and times relative to the price at some arbitrary fixed reference time τ_0. Thus $S(t) = \text{price}(\tau_0 + t) - \text{price}(\tau_0)$ is the (relative) security value at a (relative) trading time t. The desired form of $P(S,t)$ is obtained by maximizing an incomplete information-theoretic measure equivalent to the Tsallis entropy:

$$S_q = -\frac{1}{1-q}\left(1 - \int P(S,t)^q \, dS\right), \tag{1}$$

subject to constraints on three moments[2, 3, 4, 11, 12]. The resulting PDF is

$$P(S,t) = \frac{1}{Z(t)}\left\{1 + \beta(t)(q-1)[S - \overline{S}(t)]^2\right\}^{-\frac{1}{q-1}}. \tag{2}$$

Here q is a time-independent parameter indicating the degree of non-extensivity or equivalently the incompleteness of the information measure. $Z(t)$ is a normalization constant, $\overline{S}(t)$ is the mean, and $\beta(t)$ is related to the distribution's variance by

$$\sigma^2(t) = \int_{-\infty}^{\infty} [S - \overline{S}(t)]^2 P(S,t) dS = \begin{cases} \frac{1}{(5-3q)\beta(t)}, & q < \frac{5}{3} \\ \infty, & q \geq \frac{5}{3}. \end{cases} \tag{3}$$

It was shown rather unexpectedly that distributions of this non-extensive form turn out to solve a non-linear Fokker-Planck partial differential equation [2, 3, 10]

$$\frac{\partial P(S,t)}{\partial t} = -\frac{\partial}{\partial S}\left[F(S)P(S,t)\right] + \frac{D}{2}\frac{\partial^2 P(S,t)^{2-q}}{\partial S^2}. \tag{4}$$

Here $F(S) = \mu S$ is a linear driving term dependent on the market rate of return μ. Eq. (4) is solved by distributions of the Tsallis form Eq. (2) if the parameters in the latter evolve in time according to

$$\overline{S}(t) = \overline{S}(t_1)e^{\mu(t-t_1)} \tag{5}$$

$$\beta(t) = \Big\{\beta(t_1)^{-\frac{3-q}{2}}e^{\mu(3-q)(t-t_1)} + 2D\mu^{-1}(2-q)\left[\beta(t_1)Z^2(t_1)\right]^{\frac{q-1}{2}}\left[e^{\mu(3-q)(t-t_1)}-1\right]\Big\}^{-\frac{2}{3-q}} \tag{6}$$

$$Z(t)/Z(t_1) = [\beta(t)/\beta(t_1)]^{-\frac{1}{2}}. \tag{7}$$

Here t_1 is an arbitrary time; *e.g.*, it could be the shortest measured interval after the reference time τ_0, so that $t_1 = \tau_1 - \tau_0$ equals, say, one minute.

In [4] we investigated price changes in the S&P500 index. We showed that price-change distributions were well-described by distributions of the non-extensive form Eq. (2) evolving in time according to Eq. (4). The super-diffusion and fat tails characterizing this market are both a consequence of a non-extensivity parameter q greater than unity.

The nonlinear Fokker-Planck equation is a macroscopic description of how a probability distribution evolves in time. It is connected to an Ito-Langevin stochastic differential equation which describes how a particular trajectory evolves[9, 11]. The Ito-Langevin equation can be written in the general form

$$\frac{dS}{dt} = a(S,t) + b(S,t)\,\eta(t). \tag{8}$$

with a the drift coefficient and b the diffusion coefficient. In the stochastic term $\eta(t)dt = dW(t)$ is the Wiener process [9]. [$\eta(t)$ is a delta-correlated ($\langle\eta(t)\eta(t')\rangle = \delta(t-t')$), normally-distributed noise with unit variance ($\langle\eta(t)^2\rangle = 1$) and zero mean ($\langle\eta(t)\rangle = 0$).]

Eq. (4) has a corresponding Ito-Langevin equation of the form Eq. (8) with

$$a(S,t) = F(S) = \mu S, \quad b(S,t) = \sqrt{DP(S,t)^{1-q}}. \tag{9}$$

Here the Fokker-Planck equation's driving term $F(S) = \mu S$ appears as a time-independent linear drift coefficient a. The diffusion coefficient in our case is $b = \sqrt{DP(S,t)^{1-q}}$, which

exhibits explicitly at the level of the microscopic stochastic process the statistical dependence of subsequent price changes on the macroscopic PDF $P(S,t)$. That is, the memory effect representing correlations in time enters here simply via the diffusion coefficient. We argued in [4] that nonlinear Fokker-Planck time evolution can be expected in any stochastic system in which memory effects can be approximated in this simple manner as a probability-dependent diffusion coefficient.

II. DERIVATIVE PRICING FOR NON-EXTENSIVE STATISTICS

Now we turn to the question of how the Black-Scholes derivative pricing model is to be modified when the underlying security has non-extensive statistics and nonlinear Fokker-Planck dynamics.

We can define one form of portfolio $\Pi = -G + S\frac{\partial G}{\partial S}$ [8, 13, 14]. This is short one share of a derivative G and long $\partial G/\partial S$ shares of the underlying security (stock, say) S. The change in the value of the portfolio in a time dt is

$$d\Pi = -dG + dS\frac{\partial G}{\partial S}. \tag{10}$$

(The number of stock shares $\partial G/\partial S$ is of course constant during dt.) The change dG during dt is given by Ito's formula [9]. This results from Taylor expanding to first order in dt and to second in order in price change dS, and using $dW(t)^2 = dt$:

$$\begin{aligned} dG &= \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}dS^2 \\ &= \left(\frac{\partial G}{\partial t} + \frac{1}{2}b^2(S,t)\frac{\partial^2 G}{\partial S^2}\right)dt + \frac{\partial G}{\partial S}dS. \end{aligned} \tag{11}$$

Using Eq. (9), this becomes the stochastic differential equation obeyed by the derivative $G(S,t)$.

We seek a portfolio which instantaneously earns the same rate of return r as a short term risk-free security (assuming no arbitrage). Then

$$d\Pi = r\Pi dt = r\left(-G + S\frac{\partial G}{\partial S}\right)dt. \tag{12}$$

Substituting Eq. (11) into Eq. (10) and equating the result to Eq. (12) gives

$$\frac{\partial G(S,t)}{\partial t} + rS\frac{\partial G(S,t)}{\partial S} + \frac{1}{2}b^2(S,t)\frac{\partial^2 G(S,t)}{\partial S^2} = rG(S,t). \tag{13}$$

This pricing equation is analogous to the Black-Scholes result [8], generalized for an arbitrary diffusion term $b(s,t)$ [14]. The explicit dependence on the market rate of return μ has been replaced by the risk-free rate r.

There is however a difficulty hidden in Eq. (13). For our case the diffusion term $b(S,t) = \sqrt{DP^{1-q}}$ depends on the probability distribution function of the underlying stock. Hence Eq. (13) depends *implicitly* on the market's rate of return μ. To show what difficulty this entails, let us begin by reviewing the Cox and Ross approach to solving Eq. (13).

First define a two-point function $P(S,t|S',t')$ which obeys an equation very similar to Eq. (4),

$$\frac{\partial P(S,t|S',t')}{\partial t} = -\frac{\partial}{\partial S}\left[\mu S P(S,t|S',t')\right] + \frac{1}{2}\frac{\partial^2}{\partial S^2}\left[b^2(S,t)P(S,t|S',t')\right], \tag{14}$$

but with a boundary condition $P(S,t'|S',t') = \delta(S-S')$. [Here we continue to use $b = \sqrt{DP(S,t)^{1-q}}$.] The Cox-Ross solution is based on the fact that Eq. (14) is a forward Chapman-Kolmogorov equation, and as such also has a corresponding backwards form [9, 14]:

$$\frac{\partial P(S,t|S',t')}{\partial t'} = -\mu S'\frac{\partial P(S,t|S',t')}{\partial S'} - \frac{1}{2}b^2(S',t')\frac{\partial^2 P(S,t|S',t')}{\partial S'^2}. \tag{15}$$

As an example, let us seek a solution of Eq. (13) for a European-style call option $G(S,t)$ on a non-dividend-paying stock S. Following Cox and Ross, we try the form

$$G(S,t) = e^{-r(T-t)}\int G(S_T,T)\tilde{P}(S_T,T|S,t)\,dS_T. \tag{16}$$

This involves the value of G at the maturity time T. For the European call option this is

$$G(S_T,T) = \max(S_T - X, 0), \tag{17}$$

where S_T is the terminal stock price and X the exercise price. (At maturity the value of the call option is worthless if the terminal stock price is less than the exercise price; otherwise the value is the price difference.) Substituting Eq. (16) into Eq. (13), one finds that $\tilde{P}$ must solve

$$\frac{\partial \tilde{P}(S_T,T|S,t)}{\partial t} = -rS\frac{\partial \tilde{P}(S_T,T|S,t)}{\partial S} - \frac{1}{2}b^2(S,t)\frac{\partial^2 \tilde{P}(S_T,T|S,t)}{\partial S^2}, \tag{18}$$

with boundary condition $\tilde{P}(S_T,T|S,T) = \delta(S_T - S)$. This has the form of the backwards Chapman-Kolmogorov equation (15) but with μ replaced by the risk-free rate r.

In all the cases considered by Cox and Ross, the diffusion term b^2 was independent of the underlying stock's rate of return μ. Then $\tilde{P}$ was the probability distribution for a risk neutral world, and could readily be found.

Here however the diffusion term b^2 depends implicitly on the underlying stock's rate of return μ, and consequently Eq. (18) cannot be solved by assuming a risk neutral world. Assuming risk neutrality amounts to replacing μ by the risk-free rate r everywhere. Then Eq. (18) would be identical to Eq. (15) with μ replaced by r everywhere. This could be solved by a Tsallis form. Unfortunately this replacement is not justified, and we have to turn to an alternative approach.

In fact, we have found two variations on the Cox-Ross approach which permit straightforward solutions of the valuation equation for securities with non-extensive statistics. We will present both.

The first amounts to a change of variables. Define $\tilde{S} = \tilde{S}(S,t)$ as some function of S,t. Then using Ito's formula as in Eq. (11), we have

$$d\tilde{S} = \tilde{a}\,dt + \tilde{b}\,dW \tag{19}$$

where

$$\tilde{a}(\tilde{S},t) = \frac{\partial \tilde{S}}{\partial t} + \frac{1}{2}b^2\frac{\partial^2 \tilde{S}}{\partial S^2} + a\frac{\partial \tilde{S}}{\partial S}, \qquad \tilde{b}(\tilde{S},t) = b\frac{\partial \tilde{S}}{\partial S}. \tag{20}$$

Options written on $\tilde{S}$ can be evaluated for any $\tilde{b}$ which is independent of the original security's market rate of return μ. Let us consider the simplest case, where $\tilde{b} = \tilde{b}(t)$ is a function of t only. Then the second of Eqs. (20) can be solved for $\tilde{S}$:

$$\tilde{S}(S,t) = \frac{\tilde{b}(t)}{\sqrt{D}Z^{\frac{q-1}{2}}}\frac{\sinh^{-1}\left[\sqrt{\beta(q-1)}(S-\overline{S})\right]}{\sqrt{\beta(q-1)}}. \tag{21}$$

Here we have used Eqs. (9,2).

Now we consider an option $G(\tilde{S},t)$ and a portfolio $\Pi = -G + \frac{\partial G}{\partial \tilde{S}}\tilde{S}$. An analysis exactly like that leading to Eq. (13) shows that Π follows the risk-free rate of return r if

$$\frac{\partial G}{\partial t} + r\tilde{S}\frac{\partial G}{\partial \tilde{S}} + \frac{1}{2}\tilde{b}^2\frac{\partial^2 G}{\partial \tilde{S}^2} = rG. \tag{22}$$

This by construction has no dependence on μ, and hence describes a risk-free universe as in Cox and Ross [14]. In practical applications, $\tilde{S}$ can be viewed as a derivative of the original non-extensive security S, and $G(\tilde{S},t)$ is then an option involving $\tilde{S}$.

A very different route to valuing options for non-extensive securities comes from converting the market's Ito-Langevin equation into a coupled process with a constant diffusion coefficient. This uses an idea developed for time nonhomogeneous systems [9]. Consider the coupled process

$$d\hat{S} = [\mu S(t) + b(S,t)y(t)]\,dt \equiv \hat{a}\,dt \tag{23}$$

$$dy = -\gamma\, y(t)dt + \gamma\, dW(t), \tag{24}$$

where γ is a constant. Notice that $\hat{S}$ has no diffusion term; this will permit us to solve a two-variable Black-Scholes-like equation (below). First, however, we need to relate the coupled process to the original security S.

Eq. (24) is formally solved by

$$y(t) = \gamma \int_{-\infty}^{t} e^{-\gamma(t-t')}\eta(t')dt'. \tag{25}$$

In the limit $\gamma \to \infty$ this becomes a stationary, δ-correlated Gaussian process [9]. That is,

$$y(t) \to \eta(t) \text{ as } \gamma \to \infty. \tag{26}$$

Consequently as $\eta \to \infty$ Eq. (23) becomes identical to our market Ito-Langevin equation [Eq. (8)] and hence $\hat{S}$ becomes the original security S. We can then analyze Eqs. (23,24) for finite γ and take $\gamma \to \infty$ at the end.

Consider an option $G(\hat{S}, y, t)$. For finite γ, Eq. (23) has no diffusion term. Consequently to find dG we expand to first order in t and $\hat{S}$, and second order in y. The result is

$$dG = \left(\frac{\partial G}{\partial t} + \hat{a}\frac{\partial G}{\partial \hat{S}} + \frac{\gamma^2}{2}\frac{\partial^2 G}{\partial y^2}\right) dt + \frac{\partial G}{\partial y} dy. \tag{27}$$

Now construct a portfolio $\Pi = -G + \frac{\partial G}{\partial \hat{S}}\hat{S} + \frac{\partial G}{\partial y}y$. One readily finds that Π evolves at the risk-free rate r if

$$\frac{\partial G}{\partial t} + r\hat{S}\frac{\partial G}{\partial \hat{S}} + ry\frac{\partial G}{\partial y} + \frac{\gamma^2}{2}\frac{\partial^2 G}{\partial y^2} = rG. \tag{28}$$

We see that again the market rate of return μ has dropped out. Thus we have obtained a two-variable form of the usual Black-Scholes equation. Solutions can be find exactly as in Cox and Ross [14]:

$$G(\hat{S}, y, t) = e^{-r(T-t)} \int G(S_T, y_T, T)\hat{P}(S_T, y_T, T|S, y, t)\, dS_T\, dy, \tag{29}$$

where $\hat{P}$ solves the backwards equation

$$\frac{\partial \hat{P}(\hat{S}_T, y_T, T|\hat{S}, y, t)}{\partial t} = -r\hat{S}\frac{\partial \hat{P}}{\partial \hat{S}} - ry\frac{\partial \hat{P}}{\partial y} - \frac{\gamma^2}{2}\frac{\partial^2 \hat{P}}{\partial y^2}. \tag{30}$$

Practical application of this method of valuation would require solving with a finite value of γ, but a value large enough so that $y(t)$ is sufficiently close to $\eta(t)$ for time scales considered.

We can make some connection between the two approaches described above by integrating out y. One can define

$$G(\hat{S}, t) = \int dy\, G(\hat{S}, y, t) = \int d\hat{S}_T\, G(\hat{S}_T, T)\hat{P}(\hat{S}_T, T|\hat{S}, t), \tag{31}$$

where

$$\hat{P}(\hat{S}_T, T|\hat{S}, t) = \frac{1}{G(S_T, T)} \int dy\, dy_T\, G(\hat{S}_T, y_T, T)\hat{P}(\hat{S}_T, y_T, T|\hat{S}, y, t) \tag{32}$$

$$G(\hat{S}_T, T) = \int dy_T\, G(\hat{S}_T, y_T, T). \tag{33}$$

Eq. (31) is formally equivalent to the solution of Eq. (22) in the Cox-Ross form Eq. (16), in the limit $\gamma \to \infty$.

We have followed the lines of Black-Scholes and Cox-Ross to develop an options pricing formula for securities obeying non-extensive statistics and nonlinear Fokker-Planck time evolution. We showed in [4] that the S&P500 index is well described using this approach, and argued there that a description in terms of non-extensive statistics can be useful for any market in which the stylized facts of fat tails and anomalous diffusion are pronounced. The pricing formula obtained here would then be useful for options based on any such market. As an example we obtained pricing models for a European style call option on an underlying asset that pays out no dividends.

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